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FAST TRACK COMMUNICATION

Separability and entanglement of quantum states based on covariance matrices

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Online at stacks.iop.org/JPhysA/41/202002**Abstract**

We investigate the separability of quantum states based on covariance matrices. Separability criteria are presented for multipartite states. The lower bound of concurrence proposed in Vicente (2007 *Phys. Rev. A* **75** 052320) is improved by optimizing the local orthonormal observables.

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As one of the most striking features of quantum phenomena, quantum entanglement has been identified as a key non-local resource in quantum information processing such as quantum computation, quantum teleportation, dense coding, quantum cryptographic schemes, entanglement swapping and remote state preparation [1]. The study of quantum information processing has spurred a flurry of activities in the investigation of quantum entanglements. Nevertheless, despite the potential applications of quantum entangled states, the theory of quantum entanglement itself is far from being satisfied. One of the important problems in the theory of quantum entanglement is the separability: to decide whether or not a given quantum state is entangled. In principle the problem could be solved by calculating the measure of entanglement. However most proposed measures of entanglement involve extremizations which are difficult to handle analytically.

There have been some (necessary) criteria for separability, the Bell inequalities [2], PPT (positive partial transposition) [3] (which is also sufficient for the cases 2×2 and 2×3 bipartite systems [4]), reduction criterion [5, 6], majorization criterion [7], entanglement witnesses [4, 8, 9], realignment [10–12] and generalized realignment [13], as well as some necessary and sufficient operational criteria for low rank density matrices [14–16].

In [17] by using the Bloch representation of density matrices the author has presented a separability criterion that is independent of PPT and realignment criteria. It is also generalized to the multipartite case [18]. In [19] a criterion based on local uncertainty relations has been presented. It has been shown that the criterion based on local uncertainty relations is strictly stronger than the realignment criterion [20]. The covariance matrices are then introduced to solve the separability problem in [21–24]. Recently a criterion which is strictly stronger than

the realignment criterion and its nonlinear entanglement witnesses introduced in [20] has also been presented [25].

In this communication, we study the separability problem by using the covariance matrix approach. An alternative separability criterion is obtained for bipartite systems. The local orthonormal observable dependent lower bound of concurrence proposed in [22] is optimized. The covariance matrix approach is applied to multipartite systems and a set of separability criteria is obtained.

We first give a brief review of the covariance matrix criterion proposed in [22]. Let \mathcal{H}_d^A and \mathcal{H}_d^B be d -dimensional complex vector spaces, and ρ_{AB} a bipartite quantum state in $\mathcal{H}_d^A \otimes \mathcal{H}_d^B$. Let A_k (resp. B_k) be d^2 observables on \mathcal{H}_d^A (resp. \mathcal{H}_d^B) such that they form an orthonormal normalized basis of the observable space, satisfying $\text{tr}(A_k A_l) = \delta_{k,l}$ (resp. $\text{tr}(B_k B_l) = \delta_{k,l}$). Consider the total set $\{M_k\} = \{A_k \otimes I, I \otimes B_k\}$. It can be proven that [20],

$$\sum_{k=1}^{N^2} (M_k)^2 = dI, \quad \sum_{k=1}^{N^2} \langle M_k \rangle^2 = \text{tr}(\rho_{AB}^2). \tag{1}$$

The covariance matrix γ is defined with entries

$$\gamma_{ij}(\rho_{AB}, \{M_k\}) = \frac{\langle M_i M_j \rangle + \langle M_j M_i \rangle}{2} - \langle M_i \rangle \langle M_j \rangle, \tag{2}$$

which has a block structure [22]

$$\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{3}$$

where $A = \gamma(\rho_A, \{A_k\})$, $B = \gamma(\rho_B, \{B_k\})$, $C_{ij} = \langle A_i \otimes B_j \rangle_{\rho_{AB}} - \langle A_i \rangle_{\rho_A} \langle B_j \rangle_{\rho_B}$, $\rho_A = \text{Tr}_B(\rho_{AB})$, $\rho_B = \text{Tr}_A(\rho_{AB})$. Such a covariance matrix has a concavity property: for a mixed density matrix $\rho = \sum_k p_k \rho_k$ with $p_k \geq 0$ and $\sum_k p_k = 1$, one has $\gamma(\rho) \geq \sum_k p_k \gamma(\rho_k)$.

For a bipartite product state $\rho_{AB} = \rho_A \otimes \rho_B$, C in (3) is zero. Generally if ρ_{AB} is separable, then there exist states $|a_k\rangle\langle a_k|$ on \mathcal{H}_d^A , $|b_k\rangle\langle b_k|$ on \mathcal{H}_d^B and p_k such that

$$\gamma(\rho) \geq \kappa_A \oplus \kappa_B, \tag{4}$$

where $\kappa_A = \sum p_k \gamma(|a_k\rangle\langle a_k|, \{A_k\})$, $\kappa_B = \sum p_k \gamma(|b_k\rangle\langle b_k|, \{B_k\})$.

The so-called covariance matrix criterion (4) is made more efficient and physically plausible in [22]. For a separable bipartite state, it has been shown that

$$\sum_{i=1}^{d^2} |C_{ii}| \leq \frac{(1 - \text{tr}(\rho_A^2)) + (1 - \text{tr}(\rho_B^2))}{2}. \tag{5}$$

Criterion (5) depends on the choice of the orthonormal normalized basis of observable. In fact the term $\sum_{i=1}^{d^2} |C_{ii}|$ has an upper bound $\|C\|_{\text{KF}}$ which is invariant under unitary transformation and can be attained by choosing proper local orthonormal observable basis, where $\|C\|_{\text{KF}}$ stands for the Ky Fan norm of C , $\|C\|_{\text{KF}} = \text{tr} \sqrt{CC^\dagger}$, with \dagger denoting the transpose and conjugation. It has been shown in [26] that if ρ_{AB} is separable, then

$$\|C\|_{\text{KF}} \leq \frac{(1 - \text{tr}(\rho_A^2)) + (1 - \text{tr}(\rho_B^2))}{2}. \tag{6}$$

From the covariance matrix approach, we can also get an alternative criterion. From (3) and (4) we have that if ρ_{AB} is separable, then

$$X \equiv \begin{pmatrix} A - \kappa_A & C \\ C^T & B - \kappa_B \end{pmatrix} \geq 0. \tag{7}$$

Hence all the 2×2 minor submatrices of X must be positive. Namely one has

$$\begin{vmatrix} (A - \kappa_A)_{ii} & C_{ij} \\ C_{ji} & (B - \kappa_B)_{jj} \end{vmatrix} \geq 0,$$

i.e. $(A - \kappa_A)_{ii}(B - \kappa_B)_{jj} \geq C_{ij}^2$. Summing over all i, j and using (1), we get

$$\begin{aligned} \sum_{i,j=1}^d C_{i,j}^2 &\leq (\text{tr } A - \text{tr } \kappa_A)(\text{tr } B - \text{tr } \kappa_B) \\ &= (d - \text{tr } (\rho_A^2) - d + 1)(d - \text{tr } (\rho_B^2) - d + 1) \\ &= (1 - \text{tr } (\rho_A^2))(1 - \text{tr } (\rho_B^2)). \end{aligned}$$

That is

$$\|C\|_{\text{HS}}^2 \leq (1 - \text{tr } (\rho_A^2))(1 - \text{tr } (\rho_B^2)), \tag{8}$$

where $\|C\|_{\text{HS}}$ stands for the Euclid norm of C , i.e. $\|C\|_{\text{HS}} = \sqrt{\text{tr}(CC^\dagger)}$.

Formulae (6) and (8) are independent and could be complement. When

$$\sqrt{(1 - \text{tr } (\rho_A^2))(1 - \text{tr } (\rho_B^2))} < \|C\|_{\text{HS}} \leq \|C\|_{\text{KF}} \leq \frac{(1 - \text{tr } (\rho_A^2)) + (1 - \text{tr } (\rho_B^2))}{2},$$

(8) can recognize the entanglement but (6) cannot. When

$$\|C\|_{\text{HS}} \leq \sqrt{(1 - \text{tr } (\rho_A^2))(1 - \text{tr } (\rho_B^2))} \leq \frac{(1 - \text{tr } (\rho_A^2)) + (1 - \text{tr } (\rho_B^2))}{2} < \|C\|_{\text{KF}},$$

(6) can recognize the entanglement while (8) cannot.

The separability of a quantum state can also be investigated by computing the concurrence.

The concurrence of a pure state $|\psi\rangle$ is given by $C(|\psi\rangle) = \sqrt{2(1 - \text{Tr } \rho_A^2)}$ [28], where $\rho_A = \text{Tr}_B |\psi\rangle\langle\psi|$. Let ρ be a state in $\mathcal{H}_M^A \otimes \mathcal{H}_N^B$, $M \leq N$. The definition is extended to general mixed states ρ by the convex roof,

$$C(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \left\{ \sum_i p_i C(|\psi_i\rangle) : \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \right\}. \tag{9}$$

In [29] a lower bound of $C(\rho)$ has been obtained,

$$C(\rho) \geq \sqrt{\frac{2}{M(M-1)}} [\text{Max}(\|T_A(\rho)\|, \|R(\rho)\|) - 1], \tag{10}$$

where T_A and R stand for the partial transpose with respect to subsystem A and realignment, respectively. This bound is further improved based on local uncertainty relations [31],

$$C(\rho) \geq \frac{M + N - 2 - \sum_i \Delta_\rho^2(G_i^A \otimes I + I \otimes G_i^B)}{\sqrt{2M(M-1)}}, \tag{11}$$

where G_i^A and G_i^B are any set of local orthonormal observables.

Bound (11) again depends on the choice of the local orthonormal observables. In the following we show that this bound can be also optimized, in the sense that a local orthonormal observable independent up bound of the right-hand side of (11) can be obtained.

Theorem 1. Let ρ be a bipartite state in $\mathcal{H}_M^A \otimes \mathcal{H}_N^B$. $C(\rho)$ satisfies

$$C(\rho) \geq \frac{2\|C\|_{\text{KF}} - (1 - \text{Tr } \rho_A^2) - (1 - \text{Tr } \rho_B^2)}{\sqrt{2M(M-1)}}. \tag{12}$$

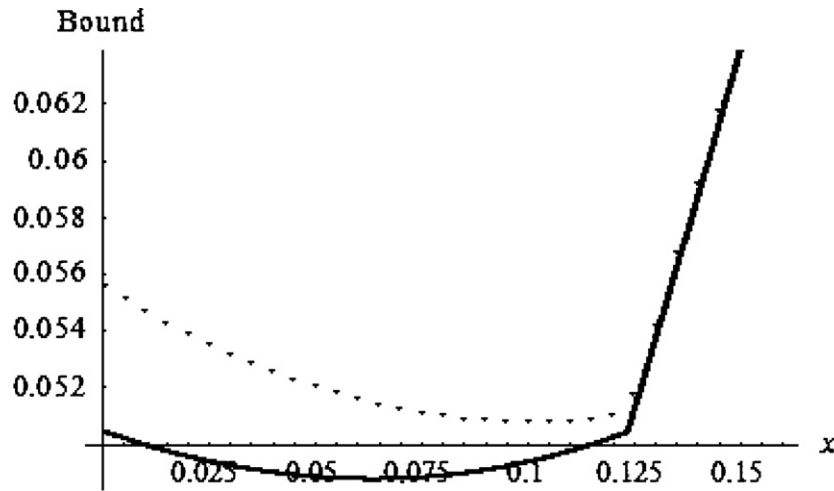


Figure 1. Lower bounds from (12) (dashed line) and (10) (solid line).

Proof. The other orthonormal normalized basis of the local orthonormal observable space can be obtained from A_i and B_i by unitary transformations U and V : $\tilde{A}_i = \sum_l U_{il} A_l$ and $\tilde{B}_j = \sum_m V_{jm}^* B_m$. Select U and V so that $C = U^\dagger \Lambda V$ is the singular value decomposition of C . Then the new observables can be written as $\tilde{A}_i = \sum_l U_{il} A_l$, $\tilde{B}_j = -\sum_m V_{jm}^* B_m$. We have

$$\begin{aligned} \sum_i \Delta_\rho^2(\tilde{A}_i \otimes I + I \otimes \tilde{B}_i) &= \sum_i [\Delta_{\rho_A}^2(\tilde{A}_i) + \Delta_{\rho_B}^2(\tilde{B}_i) + 2(\langle \tilde{A}_i \otimes \tilde{B}_i \rangle - \langle \tilde{A}_i \rangle \langle \tilde{B}_i \rangle)] \\ &= M - \text{Tr} \rho_A^2 + N - \text{Tr} \rho_B^2 - 2 \sum_i (UCV^\dagger)_{ii} \\ &= M - \text{Tr} \rho_A^2 + N - \text{Tr} \rho_B^2 - 2\|C\|_{\text{KF}}. \end{aligned}$$

Substituting the above relation into (11) we get (12). □

Bound (12) does not depend on the choice of local orthonormal observables. It can be easily applied and realized by direct measurements in experiments. It is in accord with the result in [26] where the optimization of the entanglement witness based on the local uncertainty relation has been taken into account. As an example let us consider the 3×3 bound entangled state [30],

$$\rho = \frac{1}{4} \left(I_9 - \sum_{i=0}^4 |\xi_i\rangle \langle \xi_i| \right), \tag{13}$$

where I_9 is the 9×9 identity matrix, $|\xi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle(|0\rangle - |1\rangle))$, $|\xi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle$, $|\xi_2\rangle = \frac{1}{\sqrt{2}}|2\rangle(|1\rangle - |2\rangle)$, $|\xi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)|0\rangle$, $|\xi_4\rangle = \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle)$. We simply choose the local orthonormal observables to be the normalized generators of $SU(3)$. Formula (10) gives $C(\rho) \geq 0.050$. Formula (11) gives $C(\rho) \geq 0.052$ [31], while formula (12) yields a better lower bound $C(\rho) \geq 0.0555$.

If we mix the bound entangled state (13) with $|\psi\rangle = \frac{1}{\sqrt{3}} \sum_{i=0}^2 |ii\rangle$, $\rho' = (1-x)\rho + x|\psi\rangle\langle\psi|$, it is easily seen that (12) gives a better lower bound of concurrence than formula (10) (figure 1).

The separability criteria based on the covariance matrix approach can be generalized to multipartite systems. We first consider the tripartite case, $\rho_{ABC} \in \mathcal{H}_d^A \otimes \mathcal{H}_d^B \otimes \mathcal{H}_d^C$. Take d^2 observables A_k on \mathcal{H}_A resp. B_k on \mathcal{H}_B resp. C_k on \mathcal{H}_C . Set $\{M_k\} = \{A_k \otimes I \otimes I, I \otimes B_k \otimes I, I \otimes I \otimes C_k\}$. The covariance matrix defined by (2) has then the following block structure:

$$\gamma = \begin{pmatrix} A & D & E \\ D^T & B & F \\ E^T & F^T & C \end{pmatrix}, \quad (14)$$

where $A = \gamma(\rho_A, \{A_k\})$, $B = \gamma(\rho_B, \{B_k\})$, $C = \gamma(\rho_C, \{C_k\})$, $D_{ij} = \langle A_i \otimes B_j \rangle_{\rho_{AB}} - \langle A_i \rangle_{\rho_A} \langle B_j \rangle_{\rho_B}$, $E_{ij} = \langle A_i \otimes C_j \rangle_{\rho_{AC}} - \langle A_i \rangle_{\rho_A} \langle C_j \rangle_{\rho_C}$, $F_{ij} = \langle B_i \otimes C_j \rangle_{\rho_{BC}} - \langle B_i \rangle_{\rho_B} \langle C_j \rangle_{\rho_C}$.

Theorem 2. *If ρ_{ABC} is fully separable, then*

$$\|D\|_{\text{HS}}^2 \leq (1 - \text{tr}(\rho_A^2))(1 - \text{tr}(\rho_B^2)), \quad (15)$$

$$\|E\|_{\text{HS}}^2 \leq (1 - \text{tr}(\rho_A^2))(1 - \text{tr}(\rho_C^2)), \quad (16)$$

$$\|F\|_{\text{HS}}^2 \leq (1 - \text{tr}(\rho_B^2))(1 - \text{tr}(\rho_C^2)), \quad (17)$$

and

$$2\|D\|_{\text{KF}} \leq (1 - \text{tr}(\rho_A^2)) + (1 - \text{tr}(\rho_B^2)), \quad (18)$$

$$2\|E\|_{\text{KF}} \leq (1 - \text{tr}(\rho_A^2)) + (1 - \text{tr}(\rho_C^2)), \quad (19)$$

$$2\|F\|_{\text{KF}} \leq (1 - \text{tr}(\rho_B^2)) + (1 - \text{tr}(\rho_C^2)). \quad (20)$$

Proof. For a tripartite product state $\rho_{ABC} = \rho_A \otimes \rho_B \otimes \rho_C$, D , E and F in (14) are zero. If ρ_{ABC} is fully separable, then there exist states $|a_k\rangle\langle a_k|$ in \mathcal{H}_d^A , $|b_k\rangle\langle b_k|$ in \mathcal{H}_d^B and $|c_k\rangle\langle c_k|$ in \mathcal{H}_d^C and p_k such that $\gamma(\rho) \geq \kappa_A \oplus \kappa_B \oplus \kappa_C$, where $\kappa_A = \sum p_k \gamma(|a_k\rangle\langle a_k|, \{A_k\})$, $\kappa_B = \sum p_k \gamma(|b_k\rangle\langle b_k|, \{B_k\})$ and $\kappa_C = \sum p_k \gamma(|c_k\rangle\langle c_k|, \{C_k\})$, i.e.

$$Y \equiv \begin{pmatrix} A - \kappa_A & D & E \\ D^T & B - \kappa_B & F \\ E^T & F^T & C - \kappa_C \end{pmatrix} \geq 0. \quad (21)$$

Thus all the 2×2 minor submatrices of Y must be positive. Selecting one with two rows and columns from the first two block rows and columns of Y , we have

$$\begin{vmatrix} (A - \kappa_A)_{ii} & D_{ij} \\ D_{ji} & (B - \kappa_B)_{jj} \end{vmatrix} \geq 0, \quad (22)$$

i.e. $(A - \kappa_A)_{ii}(B - \kappa_B)_{jj} \geq |D_{ij}|^2$. Summing over all i, j and using (1), we get

$$\begin{aligned} \|D\|_{\text{HS}}^2 &= \sum_{i,j=1}^{d^2} D_{i,j}^2 \leq (\text{tr} A - \text{tr} \kappa_A)(\text{tr} B - \text{tr} \kappa_B) \\ &= (d - \text{tr}(\rho_A^2) - d + 1)(d - \text{tr}(\rho_B^2) - d + 1) = (1 - \text{tr}(\rho_A^2))(1 - \text{tr}(\rho_B^2)), \end{aligned}$$

which proves (15). Equations (16) and (17) can be similarly proved.

From (22) we also have $(A - \kappa_A)_{ii} + (B - \kappa_B)_{ii} \geq 2|D_{ii}|$. Therefore,

$$\begin{aligned} \sum_i |D_{ii}| &\leq \frac{(\text{tr} A - \text{tr} \kappa_A) + (\text{tr} B - \text{tr} \kappa_B)}{2} \\ &= \frac{(d - \text{tr}(\rho_A^2) - d + 1) + (d - \text{tr}(\rho_B^2) - d + 1)}{2} \\ &= \frac{(1 - \text{tr}(\rho_A^2)) + (1 - \text{tr}(\rho_B^2))}{2}. \end{aligned} \quad (23)$$

Note that $\sum_{i=1}^{d^2} D_{ii} \leq \sum_{i=1}^{d^2} |D_{ii}|$. By using that $\text{Tr}(MU) \leq \|M\|_{\text{KF}} = \text{Tr} \sqrt{MM^\dagger}$ for any matrix M and any unitary U [27], we have $\sum_{i=1}^{d^2} D_{ii} \leq \|D\|_{\text{KF}}$.

Let $D = U^\dagger \Lambda V$ be the singular value decomposition of D . Make a transformation of the orthonormal normalized basis of the local orthonormal observable space: $\tilde{A}_i = \sum_l U_{il} A_l$ and $\tilde{B}_j = \sum_m V_{jm}^* B_m$. In the new basis we have

$$\tilde{D}_{ij} = \sum_{lm} U_{il} D_{lm} V_{jm} = (UDV^\dagger)_{ij} = \Lambda_{ij}. \quad (24)$$

Then (23) becomes

$$\sum_{i=1}^{d^2} \tilde{D}_{ii} = \|D\|_{\text{KF}} \leq \frac{(1 - \text{tr}(\rho_A^2)) + (1 - \text{tr}(\rho_B^2))}{2}$$

which proves (18). Equations (19) and (20) can be similarly treated. \square

We consider now the case that ρ_{ABC} is bi-partite separable.

Theorem 3. *If ρ_{ABC} is a bi-partite separable state with respect to the bipartite partition of the sub-systems A and BC (resp. AB and C ; resp. AC and B), then (15), (16) and (18), (19) (resp. (16), (17) and (19), (20); resp. (15), (17) and (18), (20)) must hold.*

Proof. We prove the case that ρ_{ABC} is bi-partite separable with respect to the A system and BC systems partition. The other cases can be similarly treated. In this case the matrices D and E in the covariance matrix (14) are zero. ρ_{ABC} takes the form $\rho_{ABC} = \sum_m p_m \rho_A^m \otimes \rho_{BC}^m$. Define $\kappa_A = \sum p_m \gamma(\rho_A^m, \{A_k\})$, $\kappa_{BC} = \sum p_m \gamma(\rho_{BC}^m, \{B_k \otimes I, I \otimes C_k\})$. κ_{BC} has a form

$$\kappa_{BC} = \begin{pmatrix} \kappa_B & F' \\ (F')^T & \kappa_C \end{pmatrix},$$

where $\kappa_B = \sum p_k \gamma(|b_k\rangle\langle b_k|, \{B_k\})$ and $\kappa_C = \sum p_k \gamma(|c_k\rangle\langle c_k|, \{C_k\})$, $(F')_{ij} = \sum_m p_m (\langle B_i \otimes C_j \rangle_{\rho_{BC}^m} - \langle B_i \rangle_{\rho_B^m} \langle C_j \rangle_{\rho_C^m})$. By using the concavity of the covariance matrix we have

$$\gamma(\rho_{ABC}) \geq \sum_m p_m \gamma(\rho_A^m \otimes \rho_{BC}^m) = \begin{pmatrix} \kappa_A & 0 & 0 \\ 0 & \kappa_B & F' \\ 0 & (F')^T & \kappa_C \end{pmatrix}.$$

Accounting to the method used in proving theorem 2, we get (15), (16) and (18), (19). \square

From theorem 2 and 3 we have

Corollary 1. *If two of the inequalities (15), (16) and (17) (or (18), (19) and (20)) are violated, the state must be fully entangled.*

The result of theorem 2 can be generalized to a general multipartite case $\rho \in \mathcal{H}_d^{(1)} \otimes \mathcal{H}_d^{(2)} \otimes \dots \otimes \mathcal{H}_d^{(N)}$. Define $\hat{A}_\alpha^i = I \otimes I \otimes \dots \otimes \lambda_\alpha \otimes I \otimes \dots \otimes I$, where $\lambda_0 = I/d$, λ_α ($\alpha = 1, 2, \dots, d^2 - 1$) are the normalized generators of $SU(d)$ satisfying $\text{tr}\{\lambda_\alpha \lambda_\beta\} = \delta_{\alpha\beta}$ and acting on the i th system $\mathcal{H}_d^{(i)}$, $i = 1, 2, \dots, N$. Denote $\{M_k\}$ the set of all \hat{A}_α^i . Then the covariance matrix of ρ can be written as

$$\gamma(\rho) = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \cdots & \mathcal{A}_{1N} \\ \mathcal{A}_{12}^T & \mathcal{A}_{22} \cdots & \mathcal{A}_{2N} \\ \vdots & \vdots & \vdots \\ \mathcal{A}_{1N}^T & \mathcal{A}_{2N}^T \cdots & \mathcal{A}_{NN} \end{pmatrix}, \quad (25)$$

where $\mathcal{A}_{ii} = \gamma(\rho, \{\hat{A}_k^i\})$ and $(\mathcal{A}_{ij})_{mn} = \langle \hat{A}_m^i \otimes \hat{A}_n^j \rangle - \langle \hat{A}_m^i \rangle \langle \hat{A}_n^j \rangle$ for $i \neq j$.

For a product state $\rho_{12\dots N}$, \mathcal{A}_{ij} , $i \neq j$, in (25) are zero matrices. For a fully separable mixed state, it has the form $\rho = \sum_k p_k |\psi_k^1\rangle\langle\psi_k^1| \otimes |\psi_k^2\rangle\langle\psi_k^2| \otimes \dots \otimes |\psi_k^N\rangle\langle\psi_k^N|$. Define

$$\kappa_{\mathcal{A}_{ii}} = \sum_k p_k \gamma(|\psi_k^i\rangle\langle\psi_k^i|, \{\hat{A}_i^i\}). \quad (26)$$

Then for a fully separable multipartite state ρ one has

$$Z = \begin{pmatrix} \mathcal{A}_{11} - \kappa_{\mathcal{A}_{11}} & \mathcal{A}_{12} \cdots & \mathcal{A}_{1N} \\ \mathcal{A}_{12}^T & \mathcal{A}_{22} - \kappa_{\mathcal{A}_{22}} \cdots & \mathcal{A}_{2N} \\ \vdots & \vdots & \vdots \\ \mathcal{A}_{1N}^T & \mathcal{A}_{2N}^T \cdots & \mathcal{A}_{NN} - \kappa_{\mathcal{A}_{NN}} \end{pmatrix} \geq 0. \quad (27)$$

From which we have the following separability criterion for multipartite systems:

Theorem 4. *If a state $\rho \in \mathcal{H}_d^{(1)} \otimes \mathcal{H}_d^{(2)} \otimes \dots \otimes \mathcal{H}_d^{(N)}$ is fully separable, the following inequalities*

$$\|\mathcal{A}_{ij}\|_{\text{HS}}^2 \leq (1 - \text{tr}(\rho_i^2))(1 - \text{tr}(\rho_j^2)), \quad (28)$$

$$\|\mathcal{A}_{ij}\|_{\text{KF}} \leq \frac{(1 - \text{tr}(\rho_i^2)) + (1 - \text{tr}(\rho_j^2))}{2} \quad (29)$$

must be fulfilled for any $i \neq j$.

We have studied the separability of quantum states by using the covariance matrix. An alternative separability criterion has been obtained for bipartite systems, which is a supplement of the criterion in [22]. The covariance matrix approach has been applied to multipartite systems and some related separability criteria have been obtained. The local orthonormal observable dependent lower bound of concurrence proposed in [22] has been optimized.

In dealing with the multipartite cases, we have considered that all subsystems have the same dimensions. The results can be generalized to the case that some or all subsystems have different dimensions. For instance, let $N_{\text{max}} = N_n$ be the largest dimension. One can choose N_n^2 observables \hat{A}_k^n . For other subsystems with smaller dimensions, say N_1 , one chooses N_1^2 observables \hat{A}_k^1 , $k = 1, \dots, N_1^2$, and set $\hat{A}_k^1 = 0$ for $k = N_1^2 + 1, \dots, N_n^2$.

Acknowledgments

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